THE INFLUENCE OF THE SELF-ROTATION OF CENTRAL BODIES ON THE MOVEMENTS OF THE PLANETS AND THE MOON ACCORDING TO EINSTEIN'S THEORY OF GRAVITATION

by

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In a recently published paper[42] one of us has approximated the field inside a rotating hollow sphere according to Einstein's theory of gravitation. This example seemed to be of principal interest in answering the question whether indeed, according to Einstein's theory, the rotation of distant masses produces a gravitational field equivalent to a centrifugal field. From another perspective it seems interesting now, by the same means, to perform the not too difficult task of integrating the field equations for a rotating solid sphere. In the Newtonian theory one can exactly replace the field in the space surrounding a (stationary or rotating) sphere of uniform density with the field of a material point of equal mass. Also, according to Einstein's theory the field of a resting sphere of incompressible fluid is equivalent to that of a point mass;[43] but for a rotating sphere this is not the case. In the latter case, as shown below, there appear supplementary terms corresponding to centrifugal and Coriolis forces. Since the planets find themselves in the field of the rotating sun, and the moons find themselves in the field of their respective rotating planet, it isn't precluded that one could obtain a new astronomical confirmation of Einstein's theory by observing the perturbations which result from the supplementary terms. The mathematical development carried out below, gives rise to perturbations in the orbital elements of the planets which perturbations are yet below the threshold of observation.[44] For the moons of Jupiter, however, we obtain relatively large secular perturbations which may still be hidden within the measuring error.

I. The Calculation of the $g_{\mu\nu}$ for the Field of a Rotating Solid Sphere

Symbol key:

l	radius of the hollow sphere
M	its mass
ω	its angular velocity
x', y', z'	Cartesian coordinates for a point in the integration space
x, y, z	coordinates of the reference point
k	gravitational constant
ρ_0	mean density.

The calculation will be conducted in a manner completely analogous to the work cited at the beginning of the present paper. Einstein's method of integration by approximation is used.[45] In this case, in developing the energy tensor of mass, it is assumed that the velocities of the field-inducing masses are sufficiently small compared to unity (that is, the speed of light) so that we may ignore the squares and products of their component velocities. For the example discussed in the earlier paper, this result differs in that the terms containing the centrifugal force, which are proportional to ω^2, are here omitted and that only the Coriolis terms appear. This omission is fully justified by the fact that $l\omega$ is very small for the sun and all the planets since, again, the velocity of light is 1. Moreover, in our case we consider the field at a sizable distance from the surface of the sphere, where r' signifies the distance from the center of the sphere to the integration element, and R denotes the distance from the point of reference to the integration element. We can then develop 1/R as a series involving r'/r which we shall truncate after the second order. Here r is the distance from the reference point to the sphere's center.

We proceed exactly as in the aforementioned paper, using the method suggested by Einstein.

$$\left.\begin{aligned} &g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \qquad \delta_{\mu\nu} \begin{matrix} = 1 & \mu = \nu \\ = 0 & \mu \neq \nu \end{matrix} \\ &\gamma_{\mu\nu} = \gamma'_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\sum_{a}\gamma'_{aa} \\ &\gamma'_{\mu\nu} = -\frac{k}{2\pi}\int \frac{T_{\mu\nu}(x', y', z', t - R)}{R}\, dV_0 . \end{aligned}\right\} \quad (1)$$

We next develop the energy tensor of the force-free matter:

$$T_{\mu\nu} = T^{\mu\nu} = \varrho_0 \frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = \sigma_0 \frac{dx_\mu}{dx_4}\frac{dx_\nu}{dx_4}\left(\frac{dx_4}{ds}\right)^2 \quad (2)$$

using the following expressions for the velocity components:

$$\left.\begin{aligned} \frac{dx_1}{dx_4} &= -i\frac{dx'}{dt} = ir'\omega \sin\vartheta' \sin\varphi' \\ \frac{dx_2}{dx_4} &= -i\frac{dy'}{dt} = -ir'\omega \sin\vartheta' \cos\varphi' \\ \frac{dx_3}{dx_4} &= 0 \end{aligned}\right\} \quad (3)$$

Here r', θ', ϕ' are the polar coordinates for a point in the sphere whose rotation takes place about the z-axis. Ignoring terms with ω^2, we obtain:

$$T_{\mu\nu} = \varrho_0 \left(\frac{dx_4}{ds}\right)^2 \begin{Bmatrix} 0, & 0, & 0, & ir'\omega \sin\vartheta' \sin\varphi' \\ 0, & 0, & 0, & -ir'\omega \sin\vartheta' \cos\varphi' \\ 0, & 0, & 0, & 0 \\ ir'\omega \sin\vartheta' \sin\varphi', & -ir'\omega \sin\vartheta' \cos\varphi', & 0, & 1 \end{Bmatrix} \quad (4)$$

For dV_0 we must, in accordance with equations (7) and (8) of the former paper, write:

$$dV_0 = i\frac{dx_4}{ds} r'^2 dr' \sin\vartheta'\, d\vartheta'\, d\varphi' . \quad (5)$$

For the purpose of expressing 1/R in terms of the integration variables, we choose the coordinate system in such a way that the point of reference will become situated in the X-Z plane. Then, introducing polar coordinates, we obtain:

$$x = r \sin\theta, \; y = 0, \; z = r \cos\theta$$

which yields:

$$\begin{aligned} R^2 &= (r' \sin\vartheta' \cos\varphi' - r \sin\vartheta)^2 + \\ &+ r'^2 \sin^2\vartheta' \sin^2\varphi' + (r' \cos\vartheta' - r \cos\vartheta)^2 = \\ &= r^2\left[1 - \frac{2r'}{r}(\sin\vartheta' \cos\varphi' \sin\vartheta + \right. \\ &\left. + \cos\vartheta' \cos\vartheta) + \frac{r'^2}{r^2}\right] . \end{aligned}$$

We next develop the binomial series and truncate after the second order terms:

$$\begin{aligned} \frac{1}{R} = \frac{1}{r}\Big\{ 1 &+ \frac{r'}{r}(\sin\vartheta' \cos\varphi' \sin\vartheta + \cos\vartheta' \cos\vartheta) - \\ &- \frac{1}{2}\frac{r'^2}{r^2} + \frac{3}{2}\frac{r'^2}{r^2}(\sin\vartheta' \cos\varphi' \sin\vartheta + \cos\vartheta' \cos\vartheta)^2 \Big\} . \end{aligned} \quad (6)$$

We again denote the expression within the braces by K and write:

$$1/R = K/r. \quad (6a)$$

If we now substitute equations (4), (5) and (6a) into the last line of equation (1), we obtain the following result:

$$
\begin{aligned}
\gamma'_{44} &= -\frac{i\varkappa}{2\pi}\frac{\varrho_0}{r}\int_0^l r'^2 dr'\int_0^{2\pi} d\varphi'\int_0^{\pi} d\vartheta'\left(\frac{dx_4}{ds}\right)^3 \sin\vartheta' K \\
\gamma'_{14} &= \frac{\varkappa}{2\pi}\frac{\varrho_0}{r}\,\omega\int_0^l r'^3 dr'\int_0^{2\pi} d\varphi'\int_0^{\pi} d\vartheta'\left(\frac{dx_4}{ds}\right)^3 \sin^2\vartheta' \sin\varphi' K \\
\gamma'_{24} &= -\frac{\varkappa}{2\pi}\frac{\varrho_0}{r}\,\omega\int_0^l r'^3 dr'\int_0^{2\pi} d\varphi'\int_0^{\pi} d\vartheta'\left(\frac{dx_4}{ds}\right)^3 \sin^2\vartheta' \cos\varphi' K \\
\gamma'_{11} &= \gamma'_{22} = \gamma'_{33} = \gamma'_{12} = \gamma'_{13} = \gamma'_{23} = \gamma'_{34} = 0 .
\end{aligned}
\tag{7}
$$

Ignoring terms involving ω^2, we have:

$(dx_4/ds)^3 = i$ (Cf. Eq. (11) in the former paper.)

If we substitute this expression as well as the expression for K from equations (6) and (6a) into (7), we obtain, after evaluating the integrals:

$$
\left.
\begin{aligned}
\gamma'_{44} &= \frac{\varkappa}{2\pi}\frac{M}{r} \\
\gamma'_{14} &= 0 \\
\gamma'_{24} &= -i\,\frac{\varkappa}{2\pi}\frac{M}{r}\frac{l}{5r}\,\omega l \sin\vartheta \\
\gamma'_{11} &= \gamma'_{22} = \gamma'_{33} = \gamma'_{12} = \gamma'_{13} = \gamma'_{23} = \gamma'_{34} = 0.
\end{aligned}
\right\}
\tag{8}
$$

From this it follows, according to (1), that if we reintroduce Cartesian coordinates, and replace Einstein's gravitational constant by Newton's $k = (\varkappa/(8\pi))$, then:

$$
\begin{aligned}
g_{11} &= g_{22} = g_{33} = -1 - \frac{2kM}{r} \\
g_{44} &= -1 + \frac{2kM}{r} \\
g_{24} &= -i\,\frac{4kM}{r}\frac{lx}{5r^2}\,\omega l \\
g_{12} &= g_{13} = g_{23} = g_{14} = g_{34} = 0 .
\end{aligned}
\tag{9}
$$

If we now rotate the system and so disengage ourselves from our special choice of coordinate system (where the reference point falls on the X-Z plane), then we finally obtain the coefficient matrix:

$$
g_{\mu\nu} = \left\{
\begin{matrix}
-1-\dfrac{2kM}{r} & 0, & 0, & i\dfrac{4kM}{5r}\dfrac{ly}{r^2}\omega l \\
0, & -1-\dfrac{2kM}{r} & 0, & -i\dfrac{4kM}{5r}\dfrac{lx}{r^2}\omega l \\
0, & 0 & -1-\dfrac{2kM}{r} & 0 \\
i\dfrac{4kM}{5r}\dfrac{ly}{r^2}\omega l, & -i\dfrac{4kM}{5r}\cdot\dfrac{lx}{r^2}\omega l, & 0, & -1+\dfrac{2kM}{r}.
\end{matrix}
\right.
\tag{10}
$$

II. The Equations of Motion for a Point Mass in the Field of the Rotating Solid Sphere

In what follows, we shall develop the equations of motion for a point mass in the field of a rotating solid sphere. Here we assume that the velocity of the mass is so small that we may ignore the squares and velocity products since we assume them far below the speed of light. Let it be emphasized that our objective is to find only those perturbations in the planetary motions which are caused by the rotation of the central body. To obtain a sufficiently accurate Einsteinian solution to the planetary problem, we shall have to add the terms for the well-known perihelion shift.[46] The terms which originate from the rotation of the central body follow from the first-order approximation of Einstein's theory, while the aforementioned perihelion precession was obtained from the second-order approximation. Yet we cannot take the former into account and ignore the latter. Why this is the case becomes clear from the following consideration. Those additional terms by which the effective force (developed below) differ from the Newtonian are all proportional to ωlv where v is the velocity of the planet (specifically of a satellite), while ωl is the velocity of an equatorial point on the central body. Now for the planetary system, or for a planet-moon system, the following inequality holds:

$$v > \omega l. \qquad (11)$$

If we consider terms in ωlv, then we must also consider terms in the equations of motion which contain the squares and products of the velocity components of the point mass. But if we do this, then we can no longer confine ourselves to first order in our approximations, but we must also add the second order to the Newtonian terms which are proportional to a/r (a=2kM). The square of a planet's velocity is likewise of the order a/r. Considering terms involving the squares of the velocities therefore also demands consideration of second order terms. From this it follows via inequality (11) that the calculations here might have little validity. Yet we shall, in practice, be able to use them if we remember that all the perturbations here considered are so small that we are allowed to apply them additively. We consequently reach the desired result of computing an orbit which takes into account all relativistic influences when we base the calculation on the equations used by Einstein in his work on Mercury, that is, we add in the perturbation terms computed in that work.

As was shown in the former paper, when we avail ourselves of the aforementioned approximation, using the coordinates $x_1=x$, $x_2=y$, $x_3=z$, and $x_4=it$, then the general equations of motion:

$$\frac{d^2 x_\tau}{ds^2} = \Gamma^\tau_{\mu\nu} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds}$$

become

$$\frac{d^2 x_\tau}{dt^2} = 2i\left(\Gamma^\tau_{14}\frac{dx_1}{dt} + \Gamma^\tau_{24}\frac{dx_2}{dt} + \Gamma^\tau_{34}\frac{dx_3}{dt}\right) - \Gamma^\tau_{44} \qquad (12)$$

To first order in a stationary field, the sixteen quantities $\Gamma^\tau_{\sigma 4}$ are:

$$\begin{array}{llll}
\Gamma^1_{14} = 0, & \Gamma^1_{24} = \frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_2} - \frac{\partial g_{24}}{\partial x_1}\right), & \Gamma^1_{34} = \frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_3} - \frac{\partial g_{34}}{\partial x_1}\right), & \Gamma^1_{44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_1}, \\
\Gamma^2_{14} = \frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_1} - \frac{\partial g_{14}}{\partial x_2}\right), & \Gamma^2_{24} = 0, & \Gamma^2_{34} = \frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_3} - \frac{\partial g_{34}}{\partial x_2}\right), & \Gamma^2_{44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_2}, \\
\Gamma^3_{14} = \frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_1} - \frac{\partial g_{14}}{\partial x_3}\right), & \Gamma^3_{24} = \frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_2} - \frac{\partial g_{24}}{\partial x_3}\right), & \Gamma^3_{34} = 0, & \Gamma^3_{44} - \frac{1}{2}\frac{\partial g_{44}}{\partial x_3}, \\
\Gamma^4_{14} = \frac{1}{2}\frac{\partial g_{44}}{\partial x_1}, & \Gamma^4_{24} = \frac{1}{2}\frac{\partial g_{44}}{\partial x_2}, & \Gamma^4_{34} = \frac{1}{2}\frac{\partial g_{44}}{\partial x_3}, & \Gamma^4_{44} = 0.
\end{array} \qquad (13)$$

For the field given by equation (10), this matrix becomes:

$$\begin{array}{llll}
0, & -i\frac{2kM}{5r^2}\frac{\omega l^2}{r}\frac{x^2+y^2-2z^2}{r^2}, & -i\frac{6kM}{5r^2}\frac{\omega l^2}{r}\frac{yz}{r^2}, & \frac{kM}{r^2}\frac{x}{r} \\
+i\frac{2kM}{5r^2}\frac{\omega l^2}{r}\frac{x^2+y^2-2z^2}{r^2}, & 0, & +i\frac{6kM}{5r^2}\frac{\omega l^2}{r}\frac{xz}{r^2}, & \frac{kM}{r^2}\frac{y}{r} \\
+i\frac{6kM}{5r^2}\frac{\omega l^2}{r}\frac{yz}{z^2}, & -i\frac{6kM}{5r^2}\frac{\omega l^2}{r}\frac{xz}{r^2} & 0, & \frac{kM}{r^2}\frac{z}{r} \\
-\frac{kM}{r^2}\frac{x}{r}, & -\frac{kM}{r^2}\frac{y}{r}, & -\frac{kM}{r^2}\frac{z}{r}, & 0,
\end{array} \qquad (14)$$

Substituting these values for $\Gamma^{\tau}{}_{\sigma 4}$ into equation (12) we obtain the desired equations of motion:

$$\begin{aligned}
\ddot{x} &= \frac{kM}{r^2}\frac{\omega l^2}{r}\left[\frac{4}{5}\frac{x^2+y^2-2z^2}{r^2} + \frac{12}{5}\frac{yz}{r^2}\dot{z}\right] - \frac{kM}{r^2}\frac{x}{r}, \\
\ddot{y} &= -\frac{kM}{r^2}\frac{\omega l^2}{r}\left[\frac{4}{5}\frac{x^2+y^2-2z^2}{r^2}\dot{x} + \frac{12}{5}\frac{z}{r^2}\dot{z}\right] - \frac{kM}{r^2}\frac{y}{r} \\
\ddot{z} &= \frac{kM}{r^2}\frac{\omega l^2}{r}\frac{12}{5}\frac{z}{r}\frac{x\dot{y}-y\ddot{x}}{r} - \frac{kM}{r^2}\frac{z}{r}.
\end{aligned} \qquad (15)$$

The last terms on the right-hand side represent the Newtonian force. As explained above, we have to replace them with the components of force found in Einstein's work on Mercury. The first terms on the right-hand side are those perturbation terms originating from the rotation of the central body.

III. Calculation of the Perturbations Caused by the Rotation of the Central Body

The perturbation terms in equation (15) are defined as components X, Y and Z that are caused by the rotation of the central body. We transform them into three orthogonal components S, T and W, where S shall be the radial, T the transverse, and W the orthogonal (normal to the plane of the orbit). We introduce the following symbols normally used in astronomy:

- a semi-major axis,
- e eccentricity,
- $p=a*1-e^2)$,
- $i = \angle y\Omega\Pi$ inclination of the orbit,
- $\Omega = \angle XO\Omega$ longitude of the node,
- $\omega = \angle XO\Pi$ Longitude of the pericenter,
- L_0 mean longitude at the time of epoch = the mean longitude of a planet or satellite at time t=0, (also $\angle XOP$),
- $v = \angle\Pi OP$ true anomaly,
- $u = \angle\Omega OP = v + \omega - \Omega$
- U Orbital period in days,
- $n = 2\pi/U = \sqrt{(kM/a^3)}$ = mean angular velocity
- $C = r^2\dot{v} = n\,a^2\sqrt{(1-e^2)}$ = angular momentum per unit mass.

Furthermore, for brevity we set the constant appearing in equation (15),

$$K = 4\,k\,M\,\omega l^2 / 5.$$

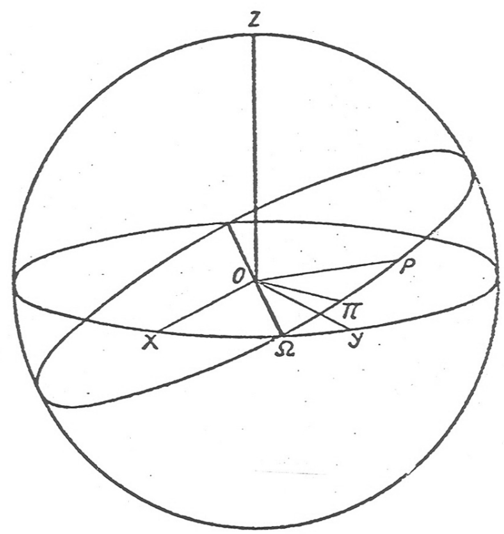

Figure 1: Π and P are the projections onto the celestial sphere of the radius vectors of the pericenter and planet respectively.

We now have:

$$\begin{aligned}
x &= r(\cos u \cos\Omega - \sin u \sin\Omega \cos i) \\
y &= r(\cos u \sin\Omega + \sin u \cos\Omega \cos i) \\
z &= r \sin u \sin i \\
r &= \frac{p}{1 + e\cos v} \\
x\dot{y} - y\ddot{x} &= C\cos i,
\end{aligned}$$

and, furthermore,

$$S = X(\cos u \cos \Omega - \sin u \sin\Omega \cos i) +$$
$$+ Y(\cos u \sin\Omega + \sin u \cos\Omega \cos i) + Z \sin u \sin i$$
$$T = -X(\sin u \cos\Omega + \cos u \sin\Omega \cos i) -$$
$$- Y(\sin u \sin\Omega - \cos u \cos\Omega \cos i) + Z \cos u \sin i$$
$$W = X \sin\Omega \sin i - Y \cos\Omega \sin i + Z \cos i.$$

Substituting these expressions for X, Y and Z from equations (15) into the formulas for S. T and W, we obtain, after much manipulation:

$$S = \frac{KC \cos i}{r^4}$$
$$T = -\frac{K\dot{r}\cos i}{r^3} = -\frac{KCe \cos i \sin v}{pr^3}$$
$$W = \frac{K \sin i}{r^4}(2C \sin u + r\dot{r}\cos u) = \tag{16}$$
$$= \frac{KC \sin i}{r^4}\left(\frac{re \sin v \cos u}{p} + 2 \sin u\right).$$

The variations in the orbital elements due to the disturbing force are given by the following equations:

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left(Se \sin v + T\frac{p}{r}\right)$$
$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left[S \sin v + T\left(e + \frac{r+a}{a}\cos v\right)\right]$$
$$\frac{di}{dt} = \frac{1}{C} Wr \cos u$$
$$\frac{d\Omega}{dt} = \frac{1}{C \sin i} Wr \sin u$$
$$\frac{d\bar{\omega}}{dt} = \frac{\sqrt{1-e^2}}{nae}\left[-S \cos v + T\left(1 + \frac{r}{p}\right)\sin v\right] +$$
$$+ 2\sin^2\frac{i}{2}\frac{d\Omega}{dt}$$
$$\frac{dL_0}{dt} = -\frac{2}{na^2}Sr + \frac{e^2}{1+\sqrt{1-e^2}}\frac{d\bar{\omega}}{dt} +$$
$$+ 2\sqrt{1-e^2}\sin^2\frac{i}{2}\frac{d\Omega}{dt},$$

After substitution of equations (16) these equations become:

$$\frac{da}{dt} = 0$$
$$\frac{de}{dt} = \frac{K \cos i}{Ca}\sin v \cdot \dot{v}$$
$$\frac{di}{dt} = \frac{K \sin i}{Cp}\cos u\,[e \sin v \cos u +$$
$$+ 2(1 + e \cos v)\sin u]\,\dot{v}$$
$$\frac{d\Omega}{dt} = \frac{K}{Cp}\sin u\,[e \sin v \cos u +$$
$$+ 2(1 + e \cos v)\sin u]\,\dot{v}$$
$$\frac{d\bar{\omega}}{dt} = -\frac{K \cos i}{Ca}\left(2 + \frac{1+e^2}{e}\cos v\right)\dot{v} +$$
$$+ 2\sin^2\frac{i}{2}\frac{d\Omega}{dt}$$
$$\frac{dL_0}{dt} = -\frac{2K \cos i}{na^2p}(1 + e \cos v)\,\dot{v} +$$
$$+ \frac{e^2}{1+\sqrt{1-e^2}}\frac{d\bar{\omega}}{dt} + 2\sqrt{1-e^2}\sin^2\frac{i}{2}\frac{d\Omega}{dt}.$$

In the customary manner for very small perturbations in the orbital elements and recalling that u=v+ω–Ω, we therefore integrate to first order and, letting $K_1=K/(na^3)$, we obtain:

$$\Delta a = 0$$
$$\Delta e = -\frac{K_1 \cos i}{\sqrt{1-e^2}}\cos v$$
$$\Delta i = -\frac{K_1 \sin i}{2(1-e^2)^{3/2}}(\cos 2u + 2e \cos v \cos^2 u)$$
$$\Delta\Omega = \frac{K_1}{(1-e^2)^{3/2}}\left[v - \frac{1}{2}\sin 2u +\right.$$
$$\left.+ e\left(\sin v - \frac{1}{2}\sin 2u \cos v\right)\right]$$
$$\Delta\bar{\omega} = -\frac{K_1 \cos i}{(1-e^2)^{3/2}}\left(2v + \frac{1+e^2}{e}\sin v\right) +$$
$$+ 2\sin^2\frac{i}{2}\Delta\Omega$$
$$\Delta L_0 = -\frac{2K_1 \cos i}{1-e^2}(v + e \sin v) +$$
$$+ \frac{e^2}{1+\sqrt{1-e^2}}\Delta\bar{\omega} + 2\sqrt{1-e^2}\sin^2\frac{i}{2}\Delta\Omega.$$

The interesting result which stems from this is that the perturbations along the semi-major axis disappear. While Δe and Δi are purely periodic, the remaining variations of the elements contain secular terms because of the relation v = nt + periodic terms. Thus

$$\Delta\Omega = \frac{K_1}{(1-e^2)^{3/2}} nt \qquad (17)$$

$$\Delta\bar{\omega} = \Delta L_0 = -\frac{2K_1}{(1-e^2)^{3/2}}\left(1 - 3\sin^2\frac{i}{2}\right)nt.$$

IV. Numerical Results

The numerical result shows that in the solar system these secular perturbations are below the observational threshold, even over the time span of a century; for they reach a maximum of 0".01 in the perihelion case of Mercury). The situation is different for the case of a planet-satellite system. Here we find a somewhat greater effect. For numerical computation it is better to transform (17). For this we propose the following notation:

- l is the radius of the planet in cm.,
- τ is the rotation period of the planet, measured in days,
- a_1 is the semi-major axis of the planet's orbit in cm.,
- U_1 is the planet's revolutionary period in days,
- U is the satellite's revolutionary period in days,
- J is the number of days in a year,
- c is the velocity of light in cm/sec,
- a is the semi-major axis of the satellite's orbit.

The formulae (17) then yield:

$$2\Delta\Omega = -\Delta\bar{\omega} = -\Delta L_0 = \frac{\pi^2 J l^2}{9c^2\tau U^2} \qquad (18)$$

which give the perturbations in the elements of the orbit of a satellite as induced by the rotation of the planet and which perturbations are in seconds of arc per century. In doing this we posited that $e^2=i^2=0$, which is consistent with the desired accuracy.

To these perturbations we must add the ones discussed by Einstein (as per section 2 of this paper) in his work on Mercury, which originate partly from the direct effect of the planet and partly because of the sun. The former are given by:

$$\Delta\Omega = 0, \quad \Delta\bar{\omega} = \Delta L_0 = \frac{5\pi^2 J}{24c^2}\frac{a^2}{U^3(1-e^2)}, \qquad (19)$$

and the latter by:[47]

$$4\Delta\Omega = \Delta\bar{\omega} = \Delta L_0 = \frac{5\pi^2 J}{12c^2}\frac{a_1^2}{U_1^3}, \qquad (20)$$

all of which are in seconds of arc per century. Here we have ignored the eccentricities and inclinations of the planetary and satellite orbits. As Table I shows, this procedure is justified because of the extreme smallness of these terms. For all the other satellites they are considerably smaller.

TABLE I

	$\Delta\Omega$	$\Delta\bar{\omega} = \Delta L_0$
Earth's Moon	+1".9	+7".7
Both Martian Satellites	+0".7	+2".7

The perturbations resulting from the rotation of the planet are listed in Table II.

TABLE II

	Jupiter			Saturn				
	V	I	II	1	2	3	4	5
$\Delta\Omega$	+1'53"	+9"	+2"	+20"	+10"	+5"	+2"	+1"
$\Delta\bar{\omega}=\Delta L_0$	−3'46"	−18"	−4"	−41"	−19"	−10"	−5"	−2"

For all other satellites the terms remain below 0".5 per century.

The largest are those terms of formulae (19) which are the Einsteinian pericentric shifts, as shown in Table III.

TABLE III

$(\Delta\Omega = 0)$

	$\Delta\bar{\omega} = \Delta L_0$		$\Delta\bar{\omega} = \Delta L_0$
Mars 1	22″	Jupiter I	4′ 28
2	2	II	1 24
Saturn 1	5′ 46″	III	26
2	3 03	IV	6
3	1 47	V	36 37
4	59	Uranus 1	22
5	25	2	10
6	3	3	3
7	2	4	1
10	2	Neptun	5

Again, for other satellites, the terms are less than 0".5 per century.

If we now add the components of the perturbation to obtain the complete relativistic effect, then we consider the correction to Newton's law as per Einstein's work on Mercury which he found resulted from a purely radial perturbation. Its components are:

$$S = -\frac{3n^2a^3C}{2c^2}\frac{\dot{v}}{r^2}, \quad T = W = 0,$$

which are independent of the choice of coordinate systems. Consequently, the corresponding perturbations (formulae (19) and Table III) may be applied to any fundamental X-Y plane. The variations of Einstein's pericentric precession are produced by a purely radial perturbation whereas the variations resulting from the central body's rotation are not purely radial

Table IV summarizes all the relativistic influences. With regard to our moon and the two satellites of Mars, only the relations (19) and (20) appear, and hence the reference plane is the planet's orbital plane. For the satellites of Jupiter and Saturn the reference plane is the planet's equatorial plane because for these planets relations (18) and (19) are required. The perturbations of the satellites or Uranus and Neptune contain only relations (19) and therefore the reference plane may be chosen as the ecliptic plane.

TABLE IV

	$\Delta\Omega$	$\Delta\bar{\omega} = \Delta L_0$	Δt
Erdmond	2″	- 8″	13,9s
Mars 1. Phobos . .	1	25	0,5
2. Deimos . .	1	5	0,4
Jupiter I	9	4′ 10″	29,5
II	2	1 20	18,9
III	0	26	12,5
IV	0	6	7,1
V	1′ 53″	32 51	1m 5,4s
Saturn 1. Mimas .	20	5 05	19,2
2. Enceladus	10	2 44	15,0
3. Tethys .	5	1 37	12,2
4. Dione .	2	54	9,2
5. Rhea . .	1	23	6,9
6. Titan . .	0	3	3,3
7. Hyperion	0	2	2,7
10. Themis .	0	2	2,9
Uranus 1. Ariel . .	0	22	3,7
2. Umbriel .	0	10	2,7
3. Titania .	0	3	1,5
4. Oberon .	0	1	1,0
Neptun . . .	0	5	2,1

With regard to the column labeled Δt, the following remark must be made. The secular variation in L_0 induces a change in the mean motion. For example. due to relativistic influences, a certain correction must be made in the time between two distinct epochs such as, e.g., eclipses of Jupiter's satellites. For one century this correction is indicated in the last column of Table IV, being obtained via the formula $\Delta t = U\Delta L_0/15$.

The variations due to the rotation of a central body are insignificant compared to the variation due to Einstein's pericentric effect. Yet they have to be taken into account for the orbits of the moons of Jupiter and Saturn. For the satellites of the outer planets, we have calculated the secular variations which derive from all the relativistic effects. Though in some instances such as the case of Jupiter's fifth satellite, they reach a considerable magnitude, yet it seems that at the present time the observations are not accurate enough to allow a proof of the theory of relativity by means of the variations in satellite orbits.